

# Backward stochastic differential equations with an unbounded generator

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## Abstract

We consider a class of backward stochastic differential equations with a possibly unbounded generator. Under a Lipschitz-type condition, we give sufficient conditions for the existence of a unique solution pair, which are weaker than the existing ones. We also give a comparison theorem as a generalisation of Peng's result.

*Keywords:* BSDEs, Unbounded generator, Existence and uniqueness, Comparison theorem.

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## 1. Introduction

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$  be a given complete filtered probability space on which a  $k$ -dimensional standard Brownian motion  $(W(t), t \geq 0)$  is defined. We assume that  $(\mathcal{F}_t, t \geq 0)$  is the augmentation of  $\sigma\{W(s) : 0 \leq s \leq t\}$  by all the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ . Consider the backward stochastic differential equation (BSDE):

$$y(t) = \xi + \int_t^T f(s, y(s), z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T], \quad (1)$$

where  $\xi$  is a given  $\mathcal{F}_T$ -measurable  $\mathbb{R}^d$ -valued random variable, and the generator  $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \rightarrow \mathbb{R}^d$  is a progressively measurable function.

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Linear equations of the type (1) were introduced by Bismut [3] in the context of stochastic linear quadratic control. The nonlinear equations (1) were introduced by Pardoux and Peng [14]. Under the global Lipschitz condition on  $f$ , i.e. under the assumption that there exists a real constant  $c > 0$  such that

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c(|y_1 - y_2| + |z_1 - z_2|), \quad (2)$$

for all  $y_1, y_2 \in \mathbb{R}^d$ ,  $z_1, z_2 \in \mathbb{R}^{d \times k}$ ,  $(t, \omega)$  a.e., they prove the existence of a unique solution pair  $(y(\cdot), z(\cdot))$ . BSDEs have been studied extensively since then, and have found wide applicability in areas such as mathematical finance, stochastic control, and stochastic controllability; see, for example, [4], [8], [9], [11], [13], [21], [17], [20], and the references therein. The global Lipschitz condition (2) has been weakened to local Lipschitz condition in [1], and to non-Lipschitz condition of a particular type in [12], [19].

The BSDEs with a possibly *unbounded* generator  $f$  are particularly important in mathematical finance. Several important interest rate models are solutions to stochastic differential equations. Such solutions are unbounded in general (see, for example, [2], [5], [23]). The problem of *market completeness* in that case gives rise to BSDEs with unbounded coefficients (see [22] for details). This has motivated [7] (see also [6]) to weaken the Lipschitz condition (2) to

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_1(t)|y_1 - y_2| + c_2(t)|z_1 - z_2|, \quad (3)$$

for all  $y_1, y_2 \in \mathbb{R}^d$ ,  $z_1, z_2 \in \mathbb{R}^{d \times k}$ ,  $(t, \omega)$  a.e., for some non-negative processes  $c_1(\cdot)$  and  $c_2(\cdot)$ . Here the processes  $c_1(\cdot)$  and  $c_2(\cdot)$  can be *unbounded*. In [7], under certain conditions on the processes  $c_1(\cdot)$  and  $c_2(\cdot)$ , the solvability of (1) is shown. The linear BSDEs with possibly unbounded coefficients are considered in [22], where only scalar equations are considered by exploiting their explicit solvability.

In this paper we also show the unique solvability of (1) that satisfies Lipschitz condition (3), but under weaker assumptions on the processes  $c_1(\cdot)$  and  $c_2(\cdot)$  as compared to [7]. Moreover, the unique solvability of (1) is shown under novel conditions on  $c_1(\cdot)$  and  $c_2(\cdot)$ , which in general are not comparable to those in [7]. A comparison theorem more general than that of Peng [15], [16], is also given.

## 2. Notation and assumptions

The following is the list of the main notations used.

- $|\cdot|$  is the Euclidian norm.
- $c_1(\cdot), c_2(\cdot)$  are given  $\mathbb{R}$ -valued progressively measurable processes.
- $\gamma(\cdot), \overline{\gamma}(\cdot)$  are given  $\mathbb{R}$ -valued positive progressively measurable processes.
- $1 < \beta_1 \in \mathbb{R}, 1 < \beta_2 \in \mathbb{R}$ , are given constants.
- $4 < \overline{\beta}_1 \in \mathbb{R}, 1 < 90\overline{\beta}_1^2/(\overline{\beta}_1^2 - 16) < \overline{\beta}_2 \in \mathbb{R}$ , are given constants.
- $\alpha_1(t) \equiv \gamma(t) + \beta_1 c_1^2(t) + \beta_2 c_2^2(t)$ ,  $\alpha_2(t) \equiv \overline{\gamma}(t) + \overline{\beta}_1 c_1(t) + \overline{\beta}_2 c_2^2(t)$ , are assumed positive.
- $p_1(t) \equiv \exp \left[ \int_0^t \alpha_1(s) ds \right]$ ,  $p_2(t) \equiv \exp \left[ \int_0^t \alpha_2(s) ds \right]$ .
- $M^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$  is the space of all  $\mathcal{F}_T$ -measurable  $\mathbb{R}^d$ -valued random variables  $\zeta$  such that  $\mathbb{E}[|\zeta|^2] < \infty$ .
- $M^2(0, T; \mathbb{R}^d)$  is the space of  $\mathcal{F}_t$ -progressively measurable  $\mathbb{R}^d$ -valued processes  $\varphi(\cdot)$  such that  $\mathbb{E} \int_0^T |\varphi(t)|^2 dt < \infty$ .
- $\widehat{M}_1^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$  (resp.  $\widehat{M}_2^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ ) is the space of all  $\mathcal{F}_T$ -measurable  $\mathbb{R}^d$ -valued random variables  $\zeta$  such that  $\mathbb{E}[p_1(T)|\zeta|^2] < \infty$  (resp.  $\mathbb{E}[p_2(T)|\zeta|^2] < \infty$ ).
- $\widehat{M}_1^2(0, T; \mathbb{R}^d)$  (resp.  $\widehat{M}_2^2(0, T; \mathbb{R}^d)$ ) is the space of  $\mathcal{F}_t$ -progressively measurable  $\mathbb{R}^d$ -valued processes  $\varphi(\cdot)$  such that  $\mathbb{E} \int_0^T p_1(t)|\varphi(t)|^2 dt < \infty$  (resp.  $\mathbb{E} \int_0^T p_2(t)|\varphi(t)|^2 dt < \infty$ ).
- $\widehat{H}_1^2(0, T; \mathbb{R}^d)$  (resp.  $\widehat{H}_2^2(0, T; \mathbb{R}^d)$ ) is the space of cadlag  $\mathcal{F}_t$ -adapted  $\mathbb{R}^d$ -valued processes  $\varphi(\cdot)$  such that  $\mathbb{E} \left[ \sup_{t \in [0, T]} p_1(t)|\varphi(t)|^2 \right] < \infty$  (resp.  $\mathbb{E} \left[ \sup_{t \in [0, T]} p_2(t)|\varphi(t)|^2 \right] < \infty$ ).

We say that the progressively measurable function  $f$  and the random variable  $\xi$ , or the pair  $(f, \xi)$ , satisfies *conditions A1* (resp. *conditions A2*) if:

- (i)  $\xi \in \widehat{M}_1^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$  (resp.  $\xi \in \widehat{M}_2^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ )
- (ii)  $|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq c_1(t)|y_1 - y_2| + c_2(t)|z_1 - z_2|$ , for all  $y_1, y_2 \in \mathbb{R}^d$ ,  $z_1, z_2 \in \mathbb{R}^{d \times k}$ ,  $(t, \omega)$  *a.e.*
- (iii)  $\left[ f(\cdot, 0, 0) \alpha_1(\cdot)^{-\frac{1}{2}} \right] \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$  (resp.  $\left[ f(\cdot, 0, 0) \alpha_2(\cdot)^{-\frac{1}{2}} \right] \in \widehat{M}_2^2(0, T; \mathbb{R}^d)$ )

The sufficient conditions for the solvability of (1), as given in [7], are similar to our conditions A2. Indeed, if we choose  $\overline{\gamma}(t) = 0$ ,  $\overline{\beta}_1 = \overline{\beta}_2 \equiv \beta$ , where  $\beta$  is *large enough*, then conditions A2 are those of [7]. Clearly, due to the process  $\overline{\gamma}(t)$  our conditions A2 are more general than those of [7]. The importance of this process is that assumption (iii) above can be suitably weakened by choosing large values for this process, which is not an option in [7]. Moreover, even if we take  $\overline{\gamma}(t) = 0$ , our assumption (i) is weaker than that of [7]. Indeed, the parameter  $\beta$  of [7] should be bigger than 446.05 (in [7] it is only claimed that this coefficient should be *large enough*, but a straightforward calculation included in our Appendix, gives this numerical lower bound). This is clearly not the case in conditions A2 where the coefficient  $\overline{\beta}_1$  is only required to be greater than 4.

The conditions A1 are new. In general, these are not comparable with conditions A2. However, in certain special cases we can compare them. For example, if  $c_1(t) = 0$ ,  $1 < \beta_2 < \overline{\beta}_2$ ,  $\gamma(t) = \overline{\gamma}(t)$ , then  $\widehat{M}_1^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d) \subset \widehat{M}_2^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ , and thus the above assumption (i) on the random variable  $\xi$  is weaker in the case of conditions A1. Similarly, if  $c_2(t) = 0$ ,  $\gamma(t) = \overline{\gamma}(t)$ ,  $\overline{\beta}_1 = 2\beta_1$ , then the above assumption (i) on the random variable  $\xi$  is weaker in the case of conditions A2.

### 3. Solvability

In this section we give sufficient conditions for the existence and uniqueness of a solution pair for (1). Our method of proof is different from [7] being based on Picard iterations, and similarly to [14], we begin with a simpler form of (1) and progress towards the general case. The proofs of the results

under conditions A1 and A2 are different and are thus given separately in most cases, but there are also similarities between them.

**Lemma 3.1.** *Let  $\phi(\cdot) \in \widehat{H}_1^2(0, T; \mathbb{R}^d)$ ,  $\psi(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$  be given, and assume that  $\sqrt{\alpha_1(\cdot)}\phi(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$ . If the pair  $(f, \xi)$  satisfies the conditions A1, then:*

(i) *there exists a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$  of equation*

$$y(t) = \xi + \int_t^T f(s, \phi(s), \psi(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T], \quad (4)$$

and  $\sqrt{\alpha_1(\cdot)}y(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$ ,

(ii) *if  $y^+(t) \equiv \mathbf{1}_{[y(t)>0]}y(t)$ , the processes*

$$\int_t^T p_1(s)y(s)z(s)dW(s) \quad \text{and} \quad \int_t^T p_1(s)y^+(s)z(s)dW(s),$$

*are martingales.*

*Proof.* (i) By making use of the Cauchy-Schwartz inequality, we first show that  $\int_0^T f(s, \phi(s), \psi(s))ds$  belongs to  $M^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ :

$$\begin{aligned} \mathbb{E} \left| \int_0^T f(s, \phi(s), \psi(s))ds \right|^2 &= \mathbb{E} \left| \int_0^T \sqrt{p_1^{-1}(s)\alpha_1(s)} \frac{\sqrt{p_1(s)}f(s, \phi(s), \psi(s))}{\sqrt{\alpha_1(s)}} ds \right|^2 \\ &\leq \mathbb{E} \left\{ \left[ \int_0^T p_1^{-1}(s)\alpha_1(s)ds \right] \left[ \int_0^T \frac{p_1(s)|f(s, \phi(s), \psi(s))|^2}{\alpha_1(s)} ds \right] \right\} \\ &\leq \mathbb{E} \int_0^T \frac{p_1(s)|f(s, \phi(s), \psi(s))|^2}{\alpha_1(s)} ds \end{aligned} \quad (5)$$

$$\begin{aligned}
&= \mathbb{E} \int_0^T \frac{p_1(s)}{\alpha_1(s)} |f(s, \phi(s), \psi(s) - f(s, 0, 0) + f(s, 0, 0)|^2 ds \\
&\leq \mathbb{E} \int_0^T \frac{p_1(s)}{\alpha_1(s)} [|f(s, \phi(s), \psi(s) - f(s, 0, 0)| + |f(s, 0, 0)|]^2 ds \\
&\leq \mathbb{E} \int_0^T \frac{p_1(s)}{\alpha_1(s)} [c_1(s)|\phi(s)| + c_2(s)|\psi(s)| + |f(s, 0, 0)|]^2 ds \\
&\leq \mathbb{E} \int_0^T \frac{p_1(s)}{\alpha_1(s)} [3c_1^2(s)|\phi(s)|^2 + 3c_2^2(s)|\psi(s)|^2 + 3|f(s, 0, 0)|^2] ds \\
&= \mathbb{E} \int_0^T \frac{3p_1(s)}{\beta_1} \frac{\beta_1 c_1^2(s)}{\gamma(s) + \beta_1 c_1^2(s) + \beta_2 c_2^2(s)} |\phi(s)|^2 ds + \frac{3p_1(s)}{\beta_2} \frac{\beta_2 c_2^2(s)}{\gamma(s) + \beta_1 c_1^2(s) + \beta_2 c_2^2(s)} |\psi(s)|^2 ds \\
&\quad + 3 \mathbb{E} \int_0^T \frac{p_1(s)|f(s, 0, 0)|^2}{\alpha_1(s)} ds \\
&\leq \frac{3}{\beta_1} \mathbb{E} \int_0^T p_1(s) |\phi(s)|^2 ds + \frac{3}{\beta_2} \mathbb{E} \int_0^T p_1(s) |\psi(s)|^2 ds + 3 \mathbb{E} \int_0^T \frac{p_1(s)|f(s, 0, 0)|^2}{\alpha_1(s)} ds < \infty
\end{aligned}$$

Since  $\xi \in \widehat{M}_1^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$  implies that  $\xi \in M^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ , it follows from Lemma 2.1 of [14] that (4) has a unique solution pair  $(y(\cdot), z(\cdot)) \in M^2(0, T; \mathbb{R}^d) \times M^2(0, T; \mathbb{R}^{d \times k})$ . Moreover, since we proved that (5) is finite, it follows from Lemma 6.2<sup>2</sup> of [7] that in fact  $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$  and  $[\sqrt{\alpha_1(\cdot)}y(\cdot)] \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$ .

(ii) The proof follows closely that in [24] (pp. 307), and since it is short, we include it here for completeness. From the Burkholder-Davis-Gundy in-

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<sup>2</sup>Note that the results in Lemma 6.2 of [7] is valid for any  $\alpha(t)$  (in the notion of [7]).

equality (see, for example, Theorem 1.5.4 in [21]), there exists a constant  $K$  such that

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t p_1(s) y(s) z(s) dW(s) \right| \right] &\leq K \mathbb{E} \left[ \int_0^T |\sqrt{p_1(s)} y(s)|^2 |\sqrt{p_1(s)} z(s)|^2 ds \right]^{\frac{1}{2}} \\ &\leq K \mathbb{E} \left[ \sup_{t \in [0, T]} |\sqrt{p_1(t)} y(t)|^2 \int_0^T |\sqrt{p_1(s)} z(s)|^2 ds \right]^{\frac{1}{2}} \\ &\leq \frac{K}{2} \mathbb{E} \left[ \sup_{t \in [0, T]} |\sqrt{p_1(t)} y(t)|^2 + \int_0^T |\sqrt{p_1(s)} z(s)|^2 ds \right] < \infty, \end{aligned}$$

where the last step follows from the fact that  $y(\cdot) \in \widehat{H}_1^2(0, T; \mathbb{R}^d)$ ,  $z(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$ , proved in part (i). The conclusion then follows from Corollary 7.22 of [10]. Since  $\sup_{t \in [0, T]} |\sqrt{p_1(t)} y^+(t)|^2 \leq \sup_{t \in [0, T]} |\sqrt{p_1(t)} y(t)|^2$ , the conclusion follows even for  $\int_0^t p_1(s) y^+(s) z(s) dW(s)$ .  $\square$

**Lemma 3.2.** *Let  $\phi(\cdot) \in \widehat{H}_2^2(0, T; \mathbb{R}^d)$ ,  $\psi(\cdot) \in \widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})$  be given, and assume that  $\sqrt{\alpha_2(\cdot)} \phi(\cdot) \in \widehat{M}_2^2(0, T; \mathbb{R}^d)$ . If the pair  $(f, \xi)$  satisfies the conditions A2, then:*

(i) *there exists a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}^d) \times \widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})$  of equation*

$$y(t) = \xi + \int_t^T f(s, \phi(s), \psi(s)) ds - \int_t^T z(s) dW(s), \quad t \in [0, T], \quad (6)$$

*and  $\sqrt{\alpha_2(\cdot)} y(\cdot) \in \widehat{M}_2^2(0, T; \mathbb{R}^d)$ ,*

(ii) *if  $y^+(t) \equiv \mathbf{1}_{[y(t) > 0]} y(t)$ , the processes*

$$\int_t^T p_2(s) y(s) z(s) dW(s) \quad \text{and} \quad \int_t^T p_2(s) y^+(s) z(s) dW(s),$$

*are martingales.*

*Proof.* The proof of part (ii) is the same as the proof of part (i) of the

previous lemma. We thus focus on part (i). We have

$$\begin{aligned}
& \mathbb{E} \left| \int_0^T f(s, \phi(s), \psi(s)) ds \right|^2 \\
& \leq \mathbb{E} \int_0^T \frac{p_2(s)}{\alpha_2(s)} [3c_1^2(s)|\phi(s)|^2 + 3c_2^2(s)|\psi(s)|^2 + 3|f(s, 0, 0)|^2] ds \\
& \leq \mathbb{E} \int_0^T \frac{3p_2(s)}{\overline{\beta_1}^2} \frac{\overline{\beta_1}c_1(s)}{\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s)} (\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s)) |\phi(s)|^2 ds \\
& \quad + \mathbb{E} \int_0^T \frac{3p_2(s)}{\overline{\beta_2}} \frac{\overline{\beta_2}c_2^2(s)}{\overline{\gamma(s)} + \overline{\beta_1}c_1(s) + \overline{\beta_2}c_2^2(s)} |\psi(s)|^2 ds + 3 \mathbb{E} \int_0^T \frac{p_2(s)|f(s, 0, 0)|^2}{\alpha_2(s)} ds \\
& \leq \frac{3}{\overline{\beta_1}^2} \mathbb{E} \int_0^T p_2(s)\alpha_2(s)|\phi(s)|^2 ds + \frac{3}{\overline{\beta_2}} \mathbb{E} \int_0^T p_2(s)|\psi(s)|^2 ds + 3 \mathbb{E} \int_0^T \frac{p_2(s)|f(s, 0, 0)|^2}{\alpha_2(s)} ds \\
& < \infty.
\end{aligned}$$

The rest of the proof is the same as in the proof of part (i) of the previous lemma.  $\square$

**Lemma 3.3.** (i) Let  $\phi(\cdot) \in \widehat{H}_1^2(0, T; \mathbb{R}^d)$  be given. If the pair  $(f, \xi)$  satisfies conditions A1, then there exists a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$  of equation

$$y(t) = \xi + \int_t^T f(s, \phi(s), z(s)) ds - \int_t^T z(s) dW(s), \quad t \in [0, T], \quad (7)$$

and  $\sqrt{\alpha_1(\cdot)}y(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$ .

(ii) Let  $\phi(\cdot) \in \widehat{H}_2^2(0, T; \mathbb{R}^d)$  be given. If the pair  $(f, \xi)$  satisfies conditions A2, then there exists a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}^d) \times$



$\widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})$  of equation

$$y(t) = \xi + \int_t^T f(s, \phi(s), z(s))ds - \int_t^T z(s)dW(s), \quad t \in [0, T],$$

and  $\sqrt{\alpha_2(\cdot)}y(\cdot) \in \widehat{M}_2^2(0, T; \mathbb{R}^d)$ .

*Proof.* (i) (*Uniqueness*) Let  $(y_1(\cdot), z_1(\cdot))$  and  $(y_2(\cdot), z_2(\cdot))$  be two solution pairs of (7) with the claimed properties. Then

$$\begin{aligned} & -dp_1(t)|y_1(t) - y_2(t)|^2 \\ & = \{-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 + 2p_1(t)(y_1(t) - y_2(t))' [f(t, \phi(t), z_1(t)) - f(t, \phi(t), z_2(t))] \\ & \quad - p_1(t)|z_1(t) - z_2(t)|^2\}dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \end{aligned} \quad (8)$$

By using the Lipschitz property of  $f$ , we have

$$\begin{aligned} & -dp_1(t)|y_1(t) - y_2(t)|^2 \\ & \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\ & \quad + 2p_1(t)|y_1(t) - y_2(t)||f(t, \phi(t), z_1(t)) - f(t, \phi(t), z_2(t))|dt \\ & \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\ & \quad + 2p_1(t)c_2(t)|y_1(t) - y_2(t)||z_1(t) - z_2(t)|dt \\ & \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\ & \quad + \beta_2 c_2^2(t)p_1(t)|y_1(t) - y_2(t)|^2dt + \beta_2^{-1}p_1(t)|z_1(t) - z_2(t)|^2dt \\ & \leq -2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t), \end{aligned}$$

which in integral form becomes

$$p_1(t)|y_1(t) - y_2(t)|^2 \leq \int_t^T -2p_1(s)(y_1(s) - y_2(s))'(z_1(s) - z_2(s))dW(s). \quad (9)$$

The stochastic integral in (9) is a local martingale that is clearly lower bounded by zero, and is thus a supermartingale (see, for example, Theorem 7.23 of [10]). Taking the expectation of both sides of (9) results in

$$\mathbb{E} [p_1(t)|y_1(t) - y_2(t)|^2] \leq -\mathbb{E} \left[ \int_t^T 2p_1(s)(y_1(s) - y_2(s))'(z_1(s) - z_2(s))dW(s) \right] \leq 0.$$

Since  $p_1(t) > 0$ , it follows that  $y_1(t) = y_2(t)$ ,  $\forall t \in [0, T]$ , a.s., which proves the uniqueness of  $y(\cdot)$ . Due to this fact, the integral form of (8) becomes

$$0 = \int_t^T p_1(s)|z_1(s) - z_2(s)|^2 ds,$$

which implies that  $z_1(t) = z_2(t)$  for *a.e.*  $t \in [0, T]$ , and thus proves the uniqueness of  $z(\cdot)$ .

(*Existence*) Let  $z_0(t) \equiv 0$ ,  $\forall t \in [0, T]$ , and for  $n \geq 1$  consider the following sequence of equations:

$$y_n(t) = \xi + \int_t^T f(s, \phi(s), z_{n-1}(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0, T]. \quad (10)$$

From Lemma 3.1 we know that these equations have unique solution pairs  $\{(y_n(\cdot), z_n(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})\}_{n \geq 1}$ , for which it also holds that  $\{\sqrt{\alpha_1(\cdot)}y_n(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^d)\}_{n \geq 1}$ . Similarly to the proof of uniqueness,

we have

$$\begin{aligned}
& -dp_1(t)|y_{n+1}(t) - y_n(t)|^2 \\
& = \{-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 + 2p_1(t)(y_{n+1}(t) - y_n(t))' [f(t, \phi(t), z_n(t)) - f(t, \phi(t), z_{n-1}(t))]\} \\
& \quad - p_1(t)|z_{n+1}(t) - z_n(t)|^2\}dt - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& \quad + 2p_1(t)|y_{n+1}(t) - y_n(t)| |f(t, \phi(t), z_n(t)) - f(t, \phi(t), z_{n-1}(t))| dt \\
& \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& \quad + 2p_1(t)c_2(t)|y_{n+1}(t) - y_n(t)||z_n(t) - z_{n-1}(t)|dt \\
& \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\
& \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\
& \quad + \beta_2 c_2^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 dt + \beta_2^{-1}p_1(t)|z_n(t) - z_{n-1}(t)|^2 dt \\
& \leq [-p_1(t)|z_{n+1}(t) - z_n(t)|^2 + \beta_2^{-1}p_1(t)|z_n(t) - z_{n-1}(t)|^2]dt \\
& \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t),
\end{aligned}$$

which in integral form becomes

$$\begin{aligned} & p_1(t)|y_{n+1}(t) - y_n(t)|^2 + \int_t^T p_1(s)|z_{n+1}(s) - z_n(s)|^2 ds \\ & \leq \beta_2^{-1} \int_t^T p_1(s)|z_n(s) - z_{n-1}(s)|^2 ds - \int_t^T 2p_1(s)(y_{n+1}(s) - y_n(s))'(z_{n+1}(s) - z_n(s))dW(s). \end{aligned}$$

From Lemma 3.1 (ii), it is clear that the stochastic integral on the right hand side is a martingale. Taking the expected values of both sides gives

$$\mathbb{E}[p_1(t)|y_{n+1}(t) - y_n(t)|^2] + \mathbb{E} \int_t^T p_1(s)|z_{n+1}(s) - z_n(s)|^2 ds \leq \beta^{-1} \mathbb{E} \int_t^T p_1(s)|z_n(s) - z_{n-1}(s)|^2 ds.$$

Let us define  $\eta_n(t) \equiv \mathbb{E} \int_t^T p_1(s)|y_n(s) - y_{n-1}(s)|^2 ds$  and  $\mu_n(t) \equiv \mathbb{E} \int_t^T p_1(s)|z_n(s) - z_{n-1}(s)|^2 ds$ . Using the same argument as in the last part of the proof of Proposition 2.2 in [14], we obtain  $\eta_{n+1}(0) \leq \beta_2^{-n} \mathbb{E} \int_0^T p_1(s)|z_1(s)|^2 ds$  and  $\mu_n(0) \leq \beta_2^{-n} \mu_1(0)$ . Since the right-hand sides of these two inequalities decrease with  $n$ , it follows that  $\{y_n\}_{n \geq 1}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^d)$ , and  $\{z_n\}_{n \geq 1}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$ . Moreover, this also implies that  $\{\sqrt{\alpha_1} y_n\}_{n \geq 1}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^d)$ . Hence, the limiting processes  $y^* = \lim_{n \rightarrow \infty} y_n$  and  $z^* = \lim_{n \rightarrow \infty} z_n$  are the solution pair of (7). In addition, when such a pair of processes is substituted in (7), then (7) becomes an example of (4) with  $\psi(\cdot) = z^*(\cdot)$ . Therefore, Lemma 3.1 applies, and we have that  $y^*(\cdot) \in \widehat{H}_1^2(0, T; \mathbb{R}^{d \times k})$ .

(ii) Due to Lemma 3.2, the proof in this case is identical to the proof of part (i) (with an obvious change of notation), and is thus omitted.  $\square$

Now we present the main result of this paper.

**Theorem 3.1.** (i) If the pair  $(f, \xi)$  satisfies conditions A1, then equation (1) has a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$ , and  $\sqrt{\alpha_1(\cdot)}y(\cdot) \in \widehat{M}_1^2(0, T; \mathbb{R}^d)$ .

(ii) If the pair  $(f, \xi)$  satisfies conditions A2, then equation (1) has a unique solution pair  $(y(\cdot), z(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}^d) \times \widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})$ , and  $\sqrt{\alpha_2(\cdot)}y(\cdot) \in \widehat{M}_2^2(0, T; \mathbb{R}^d)$ .

*Proof.* (i) (*Uniqueness*) Let  $(y_1(\cdot), z_1(\cdot))$  and  $(y_2(\cdot), z_2(\cdot))$  be two solution pairs of (1) with the claimed properties. Then we have

$$\begin{aligned}
& -dp_1(t)|y_1(t) - y_2(t)|^2 \\
& = \{-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 + 2p_1(t)(y_1(t) - y_2(t))' [f(t, y_1(t), z_1(t)) - f(t, y_2(t), z_2(t))] \\
& \quad - p_1(t)|z_1(t) - z_2(t)|^2\}dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \quad + 2p_1(t)|y_1(t) - y_2(t)||f(t, y_1(t), z_1(t)) - f(t, y_2(t), z_2(t))|dt \\
& \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \quad + 2p_1(t)|y_1(t) - y_2(t)||[c_1(t)|y_1(t) - y_2(t)| + c_2(t)|z_1(t) - z_2(t)|]dt \\
& \leq [-\alpha_1(t)p_1(t)|y_1(t) - y_2(t)|^2 - p_1(t)|z_1(t) - z_2(t)|^2]dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \quad + \beta_1 c_1^2 p_1(t)|y_1(t) - y_2(t)|^2 dt + \beta_1^{-1} p_1(t)|y_1(t) - y_2(t)|^2 dt \\
& \quad + \beta_2 c_2^2 p_1(t)|y_1(t) - y_2(t)|^2 dt + \beta_2^{-1} p_1(t)|z_1(t) - z_2(t)|^2 dt \\
& \leq [\beta_1^{-1} p_1(t)|y_1(t) - y_2(t)|^2 + (\beta_2^{-1} - 1)p_1(t)|z_1(t) - z_2(t)|^2]dt \\
& \quad - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \leq \beta_1^{-1} p_1(t)|y_1(t) - y_2(t)|^2 dt - 2p_1(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t).
\end{aligned}$$

With the help of Lemma 3.1 (ii) and the Gronwall's lemma, the conclusion follows similarly to the proof of uniqueness in Lemma 3.3.

(*Existence*) Let  $y_0(t) \equiv 0, \forall t \in [0, T]$ , and for  $n \geq 1$  consider the sequence

of equations:

$$y_n(t) = \xi + \int_t^T f(s, y_{n-1}(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0, T]. \quad (11)$$

From Lemma 3.3 we know that these equations have unique solution pairs  $\{(y_n(\cdot), z_n(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}^d) \times \widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})\}_{n \geq 1}$ . Then

$$\begin{aligned} & -dp_1(t)|y_{n+1}(t) - y_n(t)|^2 \\ & = \{-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 + 2p_1(t)(y_{n+1}(t) - y_n(t))' [f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))] \\ & \quad - p_1(t)|z_{n+1}(t) - z_n(t)|^2\}dt - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t). \\ & \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\ & \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\ & \quad + 2p_1(t)|y_{n+1}(t) - y_n(t)| |f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))| dt \\ & \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\ & \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\ & \quad + 2p_1(t)|y_{n+1}(t) - y_n(t)| [c_1(t)|y_n(t) - y_{n-1}(t)| + c_2(t)|z_{n+1}(t) - z_n(t)|] dt \\ & \leq [-\alpha_1(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 - p_1(t)|z_{n+1}(t) - z_n(t)|^2]dt \\ & \quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t))dW(t) \\ & \quad + \beta_1 c_1^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 dt + \beta_1^{-1}p_1(t)|y_n(t) - y_{n-1}(t)|^2 dt \\ & \quad + \beta_2 c_2^2(t)p_1(t)|y_{n+1}(t) - y_n(t)|^2 dt + \beta_2^{-1}p_1(t)|z_{n+1}(t) - z_n(t)|^2 dt \end{aligned}$$

$$\begin{aligned} &\leq \beta_1^{-1} p_1(t) |y_n(t) - y_{n-1}(t)|^2 dt + (\beta_2^{-1} - 1) p_1(t) |z_{n+1}(t) - z_n(t)|^2 dt \\ &\quad - 2p_1(t)(y_{n+1}(t) - y_n(t))'(z_{n+1}(t) - z_n(t)) dW(t). \end{aligned}$$

Due to Lemma 3.1 (ii), the expectation of the integral-form of this inequality becomes

$$\begin{aligned} \mathbb{E} [p_1(t) |y_{n+1}(t) - y_n(t)|^2] &\leq \mathbb{E} \int_t^T \beta_1^{-1} p_1(s) |y_n(s) - y_{n-1}(s)|^2 ds \\ &\quad + \mathbb{E} \int_t^T (\beta_2^{-1} - 1) p_1(s) |z_{n+1}(s) - z_n(s)|^2 ds, \end{aligned} \tag{12}$$

Using the notation  $\nu_{n+1}(t) \equiv \mathbb{E} \int_t^T p_1(s) |y_{n+1}(s) - y_n(s)|^2 ds$ , and similarly to the last part of the proof of Theorem 3.1 of [14], we obtain  $\nu_{n+1}(0) \leq \beta_1^{-n} \frac{1}{n!} \nu_1(0)$ . Since the sum of the right-hand side of this inequality converges, we conclude, together with (12), that  $\{y_n\}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^d)$ , and  $\{z_n\}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^{d \times k})$ . Moreover, this also implies that  $\{\sqrt{\alpha} y_n\}_{n \geq 1}$  is a Cauchy sequence in  $\widehat{M}_1^2(0, T; \mathbb{R}^d)$ . Thus the limiting processes  $y^* = \lim_{n \rightarrow \infty} y_n$  and  $z^* = \lim_{n \rightarrow \infty} z_n$  are the solution pair to (1). In addition, when such a pair of processes is substituted in (1), then (1) becomes an example of (4) with  $\phi(\cdot) = y^*(\cdot)$  and  $\psi(\cdot) = z^*(\cdot)$ . Therefore, Lemma 3.1 applies, and we have that  $y^*(\cdot) \in \widehat{H}_1^2(0, T; \mathbb{R}^{d \times k})$ .

(ii) (*Uniqueness*) Let  $(y_1(\cdot), z_1(\cdot))$  and  $(y_2(\cdot), z_2(\cdot))$  be two solution pairs of (1) with the claimed properties. Similarly to the proof of uniqueness for

part (i), we have

$$\begin{aligned}
& -dp_2(t)|y_1(t) - y_2(t)|^2 \\
& \leq [-\alpha_2(t)p_2(t)|y_1(t) - y_2(t)|^2 - p_2(t)|z_1(t) - z_2(t)|^2]dt - 2p_2(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \quad + 2p_2(t)|y_1(t) - y_2(t)|[c_1(t)|y_1(t) - y_2(t)| + c_2(t)|z_1(t) - z_2(t)|]dt \\
& \leq [-\alpha_2(t)p_2(t)|y_1(t) - y_2(t)|^2 - p_2(t)|z_1(t) - z_2(t)|^2]dt - 2p_2(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \quad + 2c_1p_2(t)|y_1(t) - y_2(t)|^2dt + \overline{\beta_2}c_2^2p_2(t)|y_1(t) - y_2(t)|^2dt + \overline{\beta_2}^{-1}p_2(t)|z_1(t) - z_2(t)|^2dt \\
& \leq (\beta_2^{-1} - 1)p_2(t)|z_1(t) - z_2(t)|^2dt - 2p_2(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t) \\
& \leq -2p_2(t)(y_1(t) - y_2(t))'(z_1(t) - z_2(t))dW(t).
\end{aligned}$$

Then the expectation of integral-form of this inequality becomes

$$\mathbb{E} [p_2(t)|y_1(t) - y_2(t)|^2] \leq \mathbb{E} \left[ \int_t^T -2p_2(s)(y_1(s) - y_2(s))'(z_1(s) - z_2(s))dW(s) \right].$$

Since the right-hand side is a martingale by Lemma 3.1 (ii), the conclusion follows.

(Existence) Let  $y_0(t) \equiv 0, \forall t \in [0, T]$ , and for  $n \geq 1$  consider the sequence of equations:

$$y_n(t) = \xi + \int_t^T f(s, y_{n-1}(s), z_n(s))ds - \int_t^T z_n(s)dW(s), \quad t \in [0, T]. \quad (13)$$

From Lemma 3.3 we know that these equations have unique solution pairs  $\{(y_n(\cdot), z_n(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}^d) \times \widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})\}_{n \geq 1}$ . By Lemma 6.2 of [7], we have following estimates:

$$\begin{aligned}
& \mathbb{E} \left[ \int_0^T p_2(t)|y_{n+1}(t) - y_n(t)|^2 \alpha_2(t)dt \right] \\
& \leq 8 \mathbb{E} \left[ \int_0^T p_2(t) \frac{|f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))|^2}{\alpha_2(t)} dt \right], \quad (14)
\end{aligned}$$



and

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt \right] \\ & \leq 45 \mathbb{E} \left[ \int_0^T p_2(t) \frac{|f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))|^2}{\alpha_2(t)} dt \right]. \end{aligned} \quad (15)$$

By the Lipschitz condition, we have

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T p_2(t) \frac{|f(t, y_n(t), z_{n+1}(t)) - f(t, y_{n-1}(t), z_n(t))|^2}{\alpha_2(t)} dt \right] \\ & \leq \mathbb{E} \int_0^T \frac{p_2(t)}{\alpha_2(t)} [c_1(t) |y_n(t) - y_{n-1}(t)| + c_2(t) |z_{n+1}(t) - z_n(t)|]^2 dt \\ & \leq 2\mathbb{E} \int_0^T \frac{p_2(t)}{\alpha_2(t)} [c_1^2(t) |y_n(t) - y_{n-1}(t)|^2 + c_2^2(t) |z_{n+1}(t) - z_n(t)|^2] dt \\ & \leq 2\mathbb{E} \int_0^T \frac{p_2(t)}{\overline{\beta_1}^2} \frac{\overline{\beta_1} c_1(t)}{\overline{\gamma(t)} + \overline{\beta_1} c_1(t) + \overline{\beta_2} c_2^2(t)} (\overline{\gamma(t)} + \overline{\beta_1} c_1(t) + \overline{\beta_2} c_2^2(t)) |y_n(t) - y_{n-1}(t)|^2 dt \\ & \quad + 2\mathbb{E} \int_0^T \frac{p_2(t)}{\overline{\beta_2}} \frac{\overline{\beta_2} c_2^2(t)}{\overline{\gamma(t)} + \overline{\beta_1} c_1(t) + \overline{\beta_2} c_2^2(t)} |z_{n+1}(t) - z_n(t)|^2 dt \\ & \leq \frac{2}{\overline{\beta_1}^2} \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt + \frac{2}{\overline{\beta_2}} \mathbb{E} \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt. \end{aligned}$$

Substituting it into (15), we have

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt \right] \\ & \leq \frac{90}{\overline{\beta_1}^2} \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt + \frac{90}{\overline{\beta_2}} \mathbb{E} \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt. \end{aligned}$$

Since  $\overline{\beta}_2 > 90$ , we have

$$\mathbb{E} \left[ \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt \right] \leq \frac{\frac{90}{\overline{\beta}_1^2}}{\left(1 - \frac{90}{\overline{\beta}_2}\right)} \mathbb{E} \int_0^T p_2(t) \alpha(t) |y_n(t) - y_{n-1}(t)|^2 dt. \quad (16)$$

Substituting it into (15), we obtain

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T p_2(t) |y_{n+1}(t) - y_n(t)|^2 \alpha_2(t) dt \right] \\ & \leq \frac{16}{\overline{\beta}_1^2} \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt + \frac{16}{\overline{\beta}_2} \mathbb{E} \int_0^T p_2(t) |z_{n+1}(t) - z_n(t)|^2 dt \\ & \leq \frac{16}{\overline{\beta}_1^2} \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt + \frac{16}{\overline{\beta}_2} \frac{\frac{90}{\overline{\beta}_1^2}}{\left(1 - \frac{90}{\overline{\beta}_2}\right)} \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt \\ & = \left[ \frac{16}{\overline{\beta}_1^2} + \frac{16}{\overline{\beta}_2} \frac{\frac{90}{\overline{\beta}_1^2}}{\left(1 - \frac{90}{\overline{\beta}_2}\right)} \right] \mathbb{E} \int_0^T p_2(t) \alpha_2(t) |y_n(t) - y_{n-1}(t)|^2 dt. \end{aligned}$$

Let  $\kappa = \left[ \frac{16}{\overline{\beta}_1^2} + \frac{16}{\overline{\beta}_2} \frac{\frac{90}{\overline{\beta}_1^2}}{\left(1 - \frac{90}{\overline{\beta}_2}\right)} \right]$ . Due to our assumptions  $\overline{\beta}_1 > 4$  and  $\overline{\beta}_2 > \frac{90\overline{\beta}_1^2}{\overline{\beta}_1^2 - 16}$ , it follows that  $\kappa < 1$ . If we define

$$\lambda_{n+1} \equiv \mathbb{E} \left[ \int_0^T p_2(t) |y_{n+1}(t) - y_n(t)|^2 \alpha_2(t) dt \right],$$

then from the above inequality we obtain

$$\lambda_{n+1} \leq \kappa \lambda_n \leq \kappa^2 \lambda_{n-1} \leq \cdots \leq \kappa^n \lambda_1.$$

This, together with (16), implies that  $\{y_n\}$  is a Cauchy sequence in  $\widehat{M}_2^2(0, T; \mathbb{R}^d)$ , and  $\{z_n\}$  is a Cauchy sequence in  $\widehat{M}_2^2(0, T; \mathbb{R}^{d \times k})$ . Moreover, this also implies

that  $\{\sqrt{\alpha}y_n\}_{n \geq 1}$  is a Cauchy sequence in  $\widehat{M}_2^2(0, T; \mathbb{R}^d)$ . Thus the limiting processes  $y^* = \lim_{n \rightarrow \infty} y_n$  and  $z^* = \lim_{n \rightarrow \infty} z_n$  are the solution pair to (1). In addition, when such a pair of processes is substituted in (1), then (1) becomes an example of (6) with  $\phi(\cdot) = y^*(\cdot)$  and  $\psi(\cdot) = z^*(\cdot)$ . Therefore, Lemma 3.2 applies, and we have that  $y^*(\cdot) \in \widehat{H}_2^2(0, T; \mathbb{R}^{d \times k})$ .  $\square$

#### 4. Comparison theorem

The following results generalise Peng's comparison theorem ([15], [16]) to equations with a possibly unbounded generator. Similarly to [15], [16], we assume that  $d = 1$ . In addition to equation (1), let us consider two further equations

$$\begin{aligned}\widehat{y}_1(t) &= \widehat{\xi}_1 + \int_t^T [\widehat{f}_1(s, \widehat{y}_1(s), \widehat{z}_1(s))]ds - \int_t^T \widehat{z}_1(s)dW(s), \quad t \in [0, T], \\ \widehat{y}_2(t) &= \widehat{\xi}_2 + \int_t^T [\widehat{f}_2(s, \widehat{y}_2(s), \widehat{z}_2(s))]ds - \int_t^T \widehat{z}_2(s)dW(s), \quad t \in [0, T].\end{aligned}$$

We assume that the pair  $(\widehat{f}_1, \widehat{\xi}_1)$  satisfies conditions A1, whereas the pair  $(\widehat{f}_2, \widehat{\xi}_2)$  satisfies conditions A2. Based on Theorem 3.1, this means that there exist unique solution pairs  $(\widehat{y}_1(\cdot), \widehat{z}_1(\cdot)) \in \widehat{H}_1^2(0, T; \mathbb{R}) \times \widehat{M}_1^2(0, T; \mathbb{R}^{1 \times k})$  and  $(\widehat{y}_2(\cdot), \widehat{z}_2(\cdot)) \in \widehat{H}_2^2(0, T; \mathbb{R}) \times \widehat{M}_2^2(0, T; \mathbb{R}^{1 \times k})$ . The following differences will appear in the proof:

$$\begin{aligned}Y_1(t) &\equiv y(t) - \widehat{y}_1(t), & Z_1(t) &\equiv z(t) - \widehat{z}_1(t), \\ Y_2(t) &\equiv y(t) - \widehat{y}_2(t), & Z_2(t) &\equiv z(t) - \widehat{z}_2(t).\end{aligned}$$

**Theorem 4.1.** (*Comparison theorem*) (i) If  $\widehat{\xi}_1 \geq \xi$  and  $\widehat{f}_1(t, y, z) \geq f(t, y, z)$ , a.s.  $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times k}$ , then  $\widehat{y}_1(t) \geq y(t)$ ,  $\forall t \in [0, T]$ , a.s..

(ii) If  $\widehat{\xi}_2 \geq \xi$  and  $\widehat{f}_2(t, y, z) \geq f(t, y, z)$ , a.s.  $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{1 \times k}$ , then  $\widehat{y}_2(t) \geq y(t)$ ,  $\forall t \in [0, T]$ , a.s..

*Proof.* (i) The equation of the difference  $Y_1(t)$  is

$$-dY_1(t) = [f(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt - Z_1(t)dW(t).$$

Denoting by  $Y_1^+(t) \equiv \mathbb{1}_{[Y_1(t)>0]}Y_1(t)$ , and using Tanaka-Meyer formula (see Theorem 6.1.2 in [18]), we obtain

$$-dY_1^+(t) = -\mathbb{1}_{[Y_1(t)>0]}dY_1(t) - \frac{1}{2}dL(t),$$

where  $L(t)$  is the local time of  $Y_1(\cdot)$  at 0. Since  $\int_0^T |Y_1(t)|dL(t) = 0$ , a.s. (see Proposition 6.1.3 in [18]), we have

$$\begin{aligned} -d[Y_1^+(t)]^2 &= 2Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt \\ &\quad - \mathbb{1}_{[Y_1(t)>0]}Z_1^2(t)dt - \mathbb{1}_{[Y_1(t)>0]}2Y_1^+(t)Z_1(t)dW(t). \end{aligned}$$

Using Itô formula, we obtain

$$\begin{aligned} &-dp_1(t)[Y_1^+(t)]^2 \\ &= -\alpha_1(t)p_1(t)[Y_1^+(t)]^2dt + 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt \\ &\quad - \mathbb{1}_{[Y_1(t)>0]}p_1(t)Z_1^2(t)dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t) \\ &\leq -\alpha_1(t)p_1(t)[Y_1^+(t)]^2dt + 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_1(t, y(t), z(t)) \\ &\quad + \widehat{f}_1(t, y(t), z(t)) - \widehat{f}_1(t, \widehat{y}_1(t), \widehat{z}_1(t))]dt \\ &\quad - \mathbb{1}_{[Y_1(t)>0]}p_1(t)Z_1^2(t)dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t) \\ &\leq [-\alpha_1(t)p_1(t)[Y_1^+(t)]^2 + 2p_1(t)Y_1^+(t)\mathbb{1}_{[Y_1(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_1(t, y(t), z(t))] \\ &\quad - \mathbb{1}_{[Y_1(t)>0]}p_1(t)Z_1^2(t)]dt - \mathbb{1}_{[Y_1(t)>0]}2p_1(t)Y_1^+(t)Z_1(t)dW(t) + \beta_1p_1(t)c_1^2(t)[Y_1^+(t)]^2dt \\ &\quad + \beta_1^{-1}p_1(t)[Y_1^+(t)]^2dt + \beta_2c_2^2(t)p_1(t)[Y_1^+(t)]^2dt + \beta_2^{-1}\mathbb{1}_{[Y_1(t)>0]}p_1(t)Z_1^2(t)dt \\ &\leq \beta_1^{-1}p_1(t)[Y_1^+(t)]^2dt - 2p_1(t)Y_1^+(t)Z_1(t)dW(t), \end{aligned}$$

which in integral form becomes

$$p_1(t)[Y_1^+(t)]^2 \leq \int_t^T \beta_1^{-1} p_1(s)[Y_1^+(s)]^2 ds - \int_t^T 2p_1(s)Y_1^+(s)Z_1(s)dW(s).$$

The stochastic integral on the right-hand side is a martingale due to Lemma 3.1 (ii). Therefore,

$$\mathbb{E}[p_1(t)[Y_1^+(t)]^2] \leq \mathbb{E} \int_t^T \beta_1^{-1} p_1(s)[Y_1^+(s)]^2 ds,$$

and the conclusion follows from Gronwall's lemma.

(ii) In a similar way to the proof of part (i), we have

$$\begin{aligned} & -dp_2(t)[Y_2^+(t)]^2 \\ &= -\alpha_2(t)p_2(t)[Y_2^+(t)]^2 dt + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_2(t, \widehat{y}_2(t), \widehat{z}_2(t))]dt \\ & \quad - \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt - 2p_2(t)Y_2^+(t)Z_2(t)dW(t) \\ & \leq -\alpha_2(t)p_2(t)[Y_2^+(t)]^2 dt + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_2(t, y(t), z(t)) \\ & \quad + \widehat{f}_2(t, y(t), z(t)) - \widehat{f}_2(t, \widehat{y}_2(t), \widehat{z}_2(t))]dt \\ & \quad - \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt - 2p_2(t)Y_2^+(t)Z_2(t)dW(t) \\ & \leq [-\alpha_2(t)p_2(t)[Y_2^+(t)]^2 + 2p_2(t)Y_2^+(t)\mathbb{1}_{[Y_2(t)>0]}[f(t, y(t), z(t)) - \widehat{f}_2(t, y(t), z(t))] \\ & \quad - \mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)]dt - \mathbb{1}_{[Y_2(t)>0]}2p_2(t)Y_2^+(t)Z_2(t)dW(t) + 2p_2(t)c_1(t)[Y_2^+(t)]^2 dt \\ & \quad + \overline{\beta}_2 c_2^2(t)p_2(t)[Y_2^+(t)]^2 dt + \overline{\beta}_2^{-1}\mathbb{1}_{[Y_2(t)>0]}p_2(t)Z_2^2(t)dt \\ & \leq -2\mathbb{1}_{[Y_2(t)>0]}p_1(t)Y_2^+(t)Z_2(t)dW(t), \end{aligned}$$

which in integral form becomes

$$p_2(t)[Y_2^+(t)]^2 \leq - \int_t^T 2\mathbb{1}_{[Y_2(s)>0]} p_2(s) Y_2^+(s) Z_2(s) dW(s).$$

Since the stochastic integral on the right-hand side is a martingale due to Lemma 3.1 (ii), we have

$$\mathbb{E}[p_2(t)[Y_2^+(t)]^2] \leq 0,$$

which concludes the proof.  $\square$

## 5. Conclusions

We have considered BSDEs with a possibly unbounded generator. Under two cases of Lipschitz-type generator, we give sufficient conditions for the existence of unique solution pairs. These are novel conditions as compared to existing ones, and are either weaker or not comparable (in general) with the existing ones. A comparison theorem is also given. It is to be expected that these results will be useful in tackling more difficult problems with unbounded generator, such as the BSDEs with a quadratic growth and the Riccati BSDE, which play a fundamental role in stochastic control.

## Appendix

Here we include the derivation of the lower bound for the parameter  $\beta$  that appears in [7]. We do so for the completeness of the paper, since in Theorem 6.1 of [7] no explicit lower bound is given, but it is only assumed that parameter  $\beta$  should be *large enough*. The notation of [7] will be used.

Equation (6.5) of [7] states that for some constants  $k$  and  $k'$  the following holds

$$\|(y, \eta)\|_\beta^2 = k\|\xi\|_\beta^2 + \frac{k'}{\beta} \left\| \frac{f}{\alpha} \right\|_\beta^2, \quad (17)$$

where the definitions of these norms are given in [7], and are just weighted Euclidian norms. From equation (5.5) of [7], which gives the definition of the norm  $\|(y, \eta)\|_\beta^2$ , and the conclusions of Lemma 6.2 of [7], we obtain

$$\begin{aligned} \|(y, \eta)\|_\beta^2 &= \|\alpha y\|_\beta^2 + \|\eta\|_\beta^2 \\ &\leq \frac{2}{\beta} \|\xi\|_\beta^2 + \frac{8}{\beta^2} \left\| \frac{f}{\alpha} \right\|_\beta^2 + 18 \|\xi\|_\beta^2 + \frac{45}{\beta} \left\| \frac{f}{\alpha} \right\|_\beta^2 \\ &= \left( 18 + \frac{2}{\beta} \right) \|\xi\|_\beta^2 + \left( \frac{45}{\beta} + \frac{8}{\beta^2} \right) \left\| \frac{f}{\alpha} \right\|_\beta^2. \end{aligned} \quad (18)$$

Comparing (17) and (18) gives  $k' = 45 + \frac{8}{\beta}$ .

Equation (6.16) of [7] states that for some constants  $\tilde{k}$  and  $\tilde{k}'$  the following holds

$$\|(\delta Y, \delta Z, \delta N)\|_\beta^2 \leq \tilde{k} \|\delta \xi\|_\beta^2 + \frac{\tilde{k}'}{\beta} \left\| \frac{\delta_2 f}{\alpha} \right\|_\beta^2. \quad (19)$$

Here, different from [7], we have used the *tilde* notation for the constants  $k$  and  $k'$  in order to avoid the clash of notation with these constants introduced in the previous paragraph. By inequality (6.5) of [7], we obtain that

$$\begin{aligned} \|(\delta Y, \delta Z, \delta N)\|_\beta^2 &\leq k \|\delta \xi\|_\beta^2 + \frac{k'}{\beta} \left\| \frac{\varphi_t}{\alpha_t^2} \right\|_\beta^2 \\ &\leq k \|\delta \xi\|_\beta^2 + \frac{3k'}{\beta} \left( \|\alpha \delta Y\|_\beta^2 + \|m^* \delta Z\|_\beta^2 + \left\| \frac{\delta_2 f}{\alpha} \right\|_\beta^2 \right) \\ &\leq k \|\delta \xi\|_\beta^2 + \frac{3k'}{\beta} \left( k \|\delta \xi\|_\beta^2 + \frac{k'}{\beta} \left\| \frac{\delta_2 f}{\alpha} \right\|_\beta^2 + \left\| \frac{\delta_2 f}{\alpha} \right\|_\beta^2 \right) \\ &= \left( k + \frac{3kk'}{\beta} \right) \|\delta \xi\|_\beta^2 + \frac{3k'}{\beta} \left( \frac{k'}{\beta} + 1 \right) \left\| \frac{\delta_2 f}{\alpha} \right\|_\beta^2. \end{aligned} \quad (20)$$

Comparing (19) and (20) gives  $\tilde{k}' = 3k' \left( \frac{k'}{\beta} + 1 \right)$ .

The inequality at the end of page 35 of [7] is

$$\|\alpha \delta Y\|_\beta^2 + \|m^* \delta Z\|_\beta^2 \leq \frac{\hat{k}'}{\beta} \|\alpha \delta y\|_\beta^2 + \|m^* \delta z\|_\beta^2, \quad (21)$$

where we have used the *hat* notation for the constant  $k'$  in order to avoid the clash of notation with this constant introduced earlier. Similarly to the previous paragraph we obtain

$$\begin{aligned} \|(\delta Y, \delta Z)\|_\beta^2 &= \|\alpha \delta Y\|_\beta^2 + \|m^* \delta Z\|_\beta^2 \leq \frac{\tilde{k}'}{\beta} \left\| \frac{\varphi}{\alpha} \right\|_\beta^2 \\ &\leq \frac{3\tilde{k}'}{\beta} \|\alpha \delta y\|_\beta^2 + \|m^* \delta z\|_\beta^2. \end{aligned} \quad (22)$$

Comparing (21) and (22) gives  $\hat{k}' = 3\tilde{k}' = 9k' \left( \frac{k'}{\beta} + 1 \right)$ . In order to apply the contraction mapping principle, it is necessary to have  $\frac{\hat{k}'}{\beta} < 1$ , i.e.

$$9 \left( \frac{45}{\beta} + \frac{8}{\beta^2} \right) \left( \frac{45}{\beta} + \frac{8}{\beta^2} + 1 \right) < 1.$$

By solving above inequality for  $\beta > 0$ , we obtain that by *large enough* in [7] it is meant that  $\beta > 446.05$ .

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